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On Fuzzy Fractional Volterra-Fredholm Model Under the Uncertainty θ-Operator of the AD Technique: Theorems and Applications

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Abstract

This article investigates the proper existence conditions and uniqueness results for a class of fuzzy fractional Caputo Volterra-Fredholm integro-differential equations (FFCV-FIDE) with initial conditions. The findings are based on Banach's contraction principle and Schaefer's fixed point theorem. Furthermore, the solution to the posed problem is found using the Adomian decomposition technique (ADT). We support the concept with several examples. The relationship between the upper and lower reduced approximation of the fuzzy solutions was demonstrated numerically and graphically using MATLAB.

Keywords: Volterra-Fredholm equation; Caputo fractional derivative; fixed point technique; ADT.

1 Introduction

Fuzzy logic is an approach to variable processing that allows for multiple possible truth values to be processed through the same variable. Fuzzy logic attempts to solve problems with an open, imprecise spectrum of data and heuristics that makes it possible to obtain an array of accurate conclusions. Fuzzy logic is designed to solve problems by considering all available information and making the best possible decision given the input [5, 6, 11].

The idea of fractional calculus (FC) dates back to the days of Leibniz and Newton. Since then, a number of mathematicians have made contributions to the theoretical development of FC. Its application to real-world issues has garnered a lot of attention in recent years [31, 33]. The mathematical modelling of systems and processes in the domains of porous media, aerodynamics, electromagnetic, physics, viscoelasticity, control theory, electro-chemistry, signal processing, chemistry, and so on gives rise to fractional differential equations (FDEs)in many engineering and scientific disciplines (see [29, 34] and the references therein). In recent years, FDEs have seen a substantial theoretical development (see [7, 20, 27] and the references therein). Some mathematicians are interested in solving problems involving integro-differential equations (IDEs). The existence and uniqueness of solutions to fractional IDEs were examined in several studies [22, 23, 35]. However, the majority of works deal with numerical analysis of fractional IDEs, or FDEs. The number of approaches for locating these approximations has increased recently. A few of these techniques are the wavelet method [25, 37], homotopy analysis method [16, 17, 21], variational iteration method [24, 32], ADT [18, 19], fractional differential transform method [10], collocation method [30], and reproducing kernel method [26].

Many academics have developed the concept of fractional IDEs in recent years. Zadeh was the first to identify the relationship between arithmetic operations and fuzzy numbers [9, 36]. Additionally, they developed the idea of fuzzy function integration. The fuzzy mapping function was also suggested by Cheng and Zadeh [28], Dubois and Prade [13], and others. Furthermore, Dubois and Prade [14] provided a basic fuzzy calculus based on the extension idea. Numerous techniques have been developed recently for solving fuzzy IDEs. Hamoud and Ghadle [15] worked on FVIDEs, which they formulated and solved using the homotopy analysis approach and variational iteration method. The residual power technique for FFCV-FIDE has been analyzed by Ahmad et al. [6]; Abu Arqub [1] discussed the kernel technique for replicating results in fuzzy Fredholm-Volterra integral equations; Alaroud et al. [8] worked on residual power series method under the generalized H-differentiability. Abu Arqub et al. [3] looked at the fractional derivative (FD) in the Atangana-Baleanu interpretation of fuzzy FDEs. Within the framework of FD Caputo-Atangana-Baleanu, they have worked on FFCV-FIDE [2]. They took advantage of the adoption of kernel functions. Adomian created the Approximation-Dynamic Technique (ADT), which has been effectively applied to solve numerous nonlinear differential equations by employing approximations that quickly converge to the intended solution in [4].

Motivated by the above works, in this study, we examine a new class of FFCV-FIDEs and demonstrate the existence and uniqueness of solutions inside their specified domain. Additionally, we will study the the approximation of solution of the following model by using ADT:

$$\begin{aligned} D^{\eta}\,\tilde{\Xi}(t,\theta) =&\tilde{\varphi}(t,\theta) + B(t)\Im(\tilde{\Xi}(t,\theta)) + \int_0^t \vartheta(t,\lambda)\Im(\tilde{\Xi}(\lambda,\theta))d\lambda \\ &+ \int_0^T \vartheta_1(t,\lambda)\Im(\tilde{\Xi}(\lambda,\theta))d\lambda, \end{aligned} \tag{1}$$

and

$$\tilde{\Xi}(0,\theta) = (\theta-1, 1-\theta), \tag{2}$$

where $t \in \Psi := (0, T]$, $\Im(\tilde{\Xi}(t,\theta))$ is a nonlinear function and kernels $\vartheta(t,\lambda)$, $\vartheta_1(t,\lambda)$ and $\tilde{\varphi}(t,\theta)$ are sufficiently smooth functions on Ψ and furthermore, $B(t) \neq 0$ on Ψ. $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1 + \beta_2 \neq 0$, and $\tilde{\beta_3} \in \mathbb{R}_\Omega$ represents $(\theta - 1, 1 - \theta)$.

An outline of the paper's structure is provided as follows. In Section 2, the fundamental concepts, notations, lemmas, and theorems of fuzzy and fuzzy FC were reviewed. In Section 3, we examine the existence and uniqueness of a solution of the given model (1)-(2). In Section 4, the solution approximation of the proposed model was tested using an ADT. Moreover, the convergence analysis is shown. Numerical experiments and a concrete computing technique are presented in Section 5.

2 Auxiliary results

The basic ideas of fuzzy calculus are defined in this part, and these definitions will be applied to the problems and approximate solutions that are put forth.

Definition 2.1. [6] $\tilde{\Xi} : \mathbb{R} \to [0,1]$ *is mapping of a fuzzy number that satisfies:* $\tilde{\Xi}$ *is upper semi-continuous on* $\mathbb{R}$*;* $\mathrm{cl}(\mathrm{supp}\,\tilde{\Xi})$ *is compact; and* $\tilde{\Xi}$ *is normal, closure, and fuzzy convex set. The set of all fuzzy numbers is represented by the symbol* $\mathbb{R}_\Omega$*. For any* $x, v \in \mathbb{R}_\Omega$ *and* $k \in \mathbb{R}$*, the addition and scalar multiplication are described by* $(x \oplus v)_\theta = x_\theta \oplus v_\theta, (k \odot x)_\theta = \left[k\underline{x}_\theta, k\bar{x}_\theta\right]$.

Definition 2.2. [8] *Let* $\tilde{\Xi}$ *is a fuzzy number in a parametric form* $\tilde{\Xi} = (\underline{\Xi}(\theta), \bar{\Xi}(\theta))$*, it fulfills the following characteristics:*

- $\underline{\Xi}(\theta)$ *be a non decreasing, bounded, and right continuous function over* $\theta \in [0,1]$.
- $\bar{\Xi}(\theta)$ *be a non increasing, bounded, and left continuous function over* $\theta \in [0,1]$.

$\underline{\Xi}(\theta) \leq \bar{\Xi}(\theta)$ *for* $\theta \in [0,1]$.

Definition 2.3. [6] *The* θ*-level set of a fuzzy number* $\tilde{\Xi} \in \mathbb{R}_\Omega$ *defined by* $[\tilde{\Xi}]_\theta$ *is identified by*

$$[\tilde{\Xi}]_\theta = \begin{cases} r \in \mathbb{R}/\tilde{\Xi}(r) \geq \theta, & \text{if} \quad 0 < \theta \leq 1, \\ \mathrm{cl}(\mathrm{supp}\,\tilde{\Xi}), & \text{if}\ \theta = 0. \end{cases}$$

The fuzzy number appears θ *level set is a bounded and closed interval* $[\underline{\Xi}(\theta), \bar{\Xi}(\theta)]$*, where* $\underline{\Xi}(\theta)$ *is the left side end point and* $\bar{\Xi}(\theta)$ *is the right side end point.*

Definition 2.4. [11] *The extended Hukuhara derivative (eH-derivative) of fuzzy-valued function* $\tilde{\Omega} : [x,y] \to \mathbb{R}_\Omega$ *at* c_0 *is expressed by*

$$\tilde{\Omega}'_{eH}(c_0) = \lim_{h \to 0} \frac{\Omega(c_0 + h) \ominus_{eH} \Omega(c_0)}{h},$$

if $\left(\tilde{\Omega}'\right)_{eH}(c_0) \in \mathbb{R}_\Omega$*, we say that* $\tilde{\Omega}$ *is extended Hukuhara differentiable (eH-differentiable) at* c_0*. Furthermore, we say that* $\tilde{\Omega}$ *is* $[(i)$*-eH*$]$*-differentiable at* c_0 *if*

$$\left(\tilde{\Omega}'_{eH}\right)_\theta(c_0) = \left[\left(\underline{\Omega_\theta}\right)'(c_0), \left(\overline{\Omega_\theta}\right)'(c_0)\right], \quad 0 \leq \theta \leq 1,$$

and that $\tilde{\Omega}$ *is* $[(ii)$*-eH*$]$*-differentiable at* c_0 *if*

$$\left(\tilde{\Omega}'_{eH}\right)_\theta(c_0) = \left[\left(\overline{\Omega_\theta}\right)'(c_0), \left(\underline{\Omega_\theta}\right)'(c_0)\right], \quad 0 \leq \theta \leq 1.$$

Definition 2.5. [11] *For a function $\Omega(t)$, the fractional integral (FI) of order $\eta > 0$ in the Riemann-Liouville (R-L) sense is defined as follows:*

$$I^{\eta}\Omega(u) = \frac{1}{\Gamma(\eta)}\int_0^u (u-t)^{\eta-1}\Omega(t)dt, \quad u>0, \quad \eta\in\mathbb{R},$$
$$I^0\Omega(u) = \Omega(u).$$

Definition 2.6. [11] *Based on its θ-level examples, the R-L FI of order η of the fuzzy function $\tilde{\Omega}(u,\eta)$ may be expressed as follows:*

$$\left[I^{\eta}\tilde{\Omega}(u;\theta)\right] = \left[I^{\eta}\underline{\Omega}(u;\theta), I^{\eta}\bar{\Omega}(u;\theta)\right],$$

where

$$I^{\eta}\underline{\Omega}(u,\theta) = \frac{1}{\Gamma(\eta)}\int_0^u (u-t)^{\eta-1}\underline{\Omega}(t,\theta)dt, \quad u>0, \quad \eta\in\mathbb{R},$$
$$I^{\eta}\bar{\Omega}(u,\theta) = \frac{1}{\Gamma(\eta)}\int_0^u (u-t)^{\eta-1}\bar{\Omega}(t,\theta)dt, \quad u>0, \quad \eta\in\mathbb{R}.$$

Definition 2.7. [12] *The order η Caputo FD is expressed in the following equation:*

$$D^{\eta}\Omega(u) = \begin{cases} \dfrac{1}{\Gamma(n-\eta)}\displaystyle\int_0^u (u-t)^{n-\eta-1}\Omega^n(t)dt, & n-1\le\eta<n, \\ \dfrac{d^n}{dt^n}\Omega(u), & \eta=n, n\in\mathbb{N}. \end{cases}$$

The characteristics

- $I^{\eta_1}I^{\eta_2}\Omega(u) = I^{\eta_1+\eta_2}\Omega(u), \quad \eta_1,\eta_2>0,$
- $I^{\eta_1}\left(u^{\eta_2}\right) = \left\{\frac{\Gamma(\eta_2+1)u_1+\eta_2}{\Gamma(\eta_2+\eta_1+1)}, \quad \eta_2>0, \eta_1>-1, u>0.\right.$

Definition 2.8. [11] *Based on its θ-level examples, the Caputo FD of order η of the fuzzy function $\tilde{\Omega}(u,\theta)$ may be expressed as follows:*

$$\left[D^{\eta}\tilde{\Omega}(u;\theta)\right] = \left[D^{\eta}\underline{\Omega}(u;\theta), D^{\eta}\bar{\Omega}(u;\theta)\right],$$

where,

$$D^{\eta}\underline{\Omega}(u,\theta) = \begin{cases} \dfrac{1}{\Gamma(m-\eta)}\displaystyle\int_0^u (u-t)^{m-\eta-1}\underline{\Omega}^m(u,\theta)dt, & m-1\le\eta<m, \\ \dfrac{d^m}{dt^m}\underline{\Omega}(u,\theta), & \eta=m, m\in\mathbb{N}, \end{cases}$$

$$D^{\eta}[\bar{\Omega}(u,\theta)] = \begin{cases} \dfrac{1}{\Gamma(m-\eta)}\displaystyle\int_0^u (u-t)^{m-\eta-1}\bar{\Omega}^m(u,\theta)dt, & m-1\le\eta<m, \\ \dfrac{d^m}{dt^m}\bar{\Omega}(u,\theta), & \eta=m, m\in\mathbb{N}. \end{cases}$$

Note $\tilde{\Omega}_{\theta}(t)$ be shown as $\tilde{\Omega}(t,\theta)$.

3 Existence and uniqueness results

We will discuss the existence and uniqueness results for for FFCV-FIDE (1) in this section. To facilitate understanding, we have included a list of the theories that we will use to deepen our discussion.

(H1) The function $\Im(\tilde{\Xi}(t,\theta))$ is satisfies the Lipschitz condition with respect to $\tilde{\Xi}(t,\theta)$, with $L(>0)$ is Lipschitz constant, and $\Im(0)=0,\ \forall\, t\in\Psi$.

(H2) The kernels $\vartheta(t,\lambda),\vartheta_1(t,\lambda)$ are bounded and continuous by $\Theta_1>0$ and $\Theta_1^*>0$ on $\Psi\times\Psi$.

(H3) The functions $B(t)$ and $\tilde{\varphi}(t,\theta)$ are bounded by $\Theta_2(>0)$ and $\Theta_3(>0)$ respectively.

Theorem 3.1. *The FFCV-FIDE (1)-(2) has a unique solution under the hypotheses (H1)-(H3),* $\forall\, t\in\Psi$, *if*

$$(\Theta_2(\eta+1)+\Theta_1+\Theta_1^*)\,L<\Gamma(\eta+2),$$

is satisfied.

Proof. Applying I^η to both sides of (1) yields

$$\begin{aligned}\tilde{\Xi}(t,\theta)=&\tilde{\Xi}_0+I^\eta\tilde{\varphi}(t,\theta)+I^\eta(B(t)\Im\tilde{\Xi}(t,\theta))\\&+I^\eta\left[\int_0^t\vartheta(t,\lambda)\Im(\tilde{\Xi}(\lambda,\theta))d\lambda+\int_0^T\vartheta_1(t,\lambda)_1\Im(\tilde{\Xi}(\lambda,\theta))d\lambda\right].\end{aligned}\tag{3}$$

The above equation is now written as follows:

$$\Lambda\tilde{\Xi}(t,\theta)=\tilde{\Xi}(t,\theta),$$

where the operator Λ is given as

$$\begin{aligned}\Lambda\tilde{\Xi}(t,\theta)=&\tilde{\Xi}_0+I^\eta\tilde{\varphi}(t,\theta)+I^\eta(B(t)\Im(\tilde{\Xi}(t,\theta)))\\&+I^\eta\left[\int_0^t\vartheta(t,\lambda)\Im(\tilde{\Xi}(\lambda,\theta))d\lambda+\int_0^T\vartheta_1(t,\lambda)_1\Im(\tilde{\Xi}(\lambda,\theta))d\lambda\right].\end{aligned}$$

Let $\tilde{\Xi}_1(t,\theta), \tilde{\Xi}_2(t,\theta) \in C[0,T]$. Hence, $\forall\ t \in [0,T]$, we get

$$\begin{aligned}
\left|\Lambda\tilde{\Xi}_1(t,\theta) - \Lambda\tilde{\Xi}_2(t,\theta)\right| \leq & I^\eta \left(|B(t)| \left|\Im\left(\tilde{\Xi}_1(t,\theta)\right) - \Im\left(\tilde{\Xi}_2(t,\theta)\right)\right|\right) \\
& + I^\eta \left[\int_0^t |\vartheta(t,\lambda)| \left|\Im\left(\tilde{\Xi}_1(\lambda,\theta)\right) - \Im\left(\tilde{\Xi}_2(\lambda,\theta)\right)\right| d\lambda\right] \\
& + I^\eta \left[\int_0^T |\vartheta_1(t,\lambda)| \left|\Im\left(\tilde{\Xi}_1(\lambda,\theta)\right) - \Im\left(\tilde{\Xi}_2(\lambda,\theta)\right)\right| d\lambda\right] \\
\leq & \frac{\Theta_2 L}{\Gamma(\eta)} \int_0^t (t-\lambda)^{\eta-1} \left|\tilde{\Xi}_1(\lambda,\theta) - \tilde{\Xi}_2(\lambda,\theta)\right| d\lambda \\
& + \frac{\Theta_1 L}{\Gamma(\eta)} \int_0^t (t-\lambda)^{\eta-1} \left[\int_0^\lambda \left|\tilde{\Xi}_1(s,\theta) - \tilde{\Xi}_2(s,\theta)\right| ds\right] d\lambda \\
& + \frac{\Theta_1^* L}{\Gamma(\eta)} \int_0^t (t-\lambda)^{\eta-1} \left[\int_0^T \left|\tilde{\Xi}_1(s,\theta) - \tilde{\Xi}_2(s,\theta)\right| ds\right] d\lambda \\
\leq & \left(\frac{(\Theta_2(\eta+1) + \Theta_1 + \Theta_1^*) L}{\Gamma(\eta+2)}\right) \left\|\tilde{\Xi}_1 - \tilde{\Xi}_2\right\|.
\end{aligned}$$

This suggests

$$\left\|\Lambda\tilde{\Xi}_1(t,\theta) - \Lambda\tilde{\Xi}_2(t,\theta)\right\| \leq \left(\frac{(\Theta_2(\eta+1) + \Theta_1 + \Theta_1^*) L}{\Gamma(\eta+2)}\right) \left\|\tilde{\Xi}_1 - \tilde{\Xi}_2\right\|,$$

where

$$\because\ \Upsilon := \frac{(\Theta_2(\eta+1) + \Theta_1 + \Theta_1^*) L}{\Gamma(\eta+2)} < 1. \tag{4}$$

Be aware that the space $(C[0,T], \|.\|)$ is a Banach space. Consequently, we may infer that the FFCV-FIDE (1)-(2) has a unique solution when using the Banach contraction principle.

Theorem 3.2. *Let (H1)-(H3) contains true hypotheses. Furthermore, we consider that* $|\Im(\tilde{\Xi}(t,\theta))| \leq \Theta^*$, $\forall\, t \in [0,T]$ *and* $\tilde{\Xi}(t,\theta) \in \mathbb{R}_\Omega$. *The FFCV-FIDE (1)-(2) has at least one solution in* Ψ.

Proof. Let $\Phi : C([0,T], \mathbb{R}_\Omega) \to C([0,T], \mathbb{R}_\Omega)$ be the operator defined by

$$\begin{aligned}
\Phi(\tilde{\Xi}(t,\theta)) = & \frac{1}{\Gamma(\eta)} \int_0^T (T-\lambda)^{\eta-1} \Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta)) \\
& + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big] d\lambda \\
& + \frac{1}{\Gamma(\eta)} \int_0^t (t-\lambda)^{\eta-1} \Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds \\
& + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big] d\lambda.
\end{aligned}$$

Here, we demonstrate the fixed point of the operator Φ by demonstrating that Φ is continuous on $C([0,T], \mathbb{R}_\Omega)$ and compact on each bounded subset of $C([0,T], \mathbb{R}_\Omega)$. This implies that the statement in part one of Schaefer's theorem is false, which in turn implies that part two of Schaefer's theorem must be true, as will be demonstrated through a series of steps.

(i) Initially, we establish the continuity of the operator Φ. Let $\tilde{\Xi}_n$ be a sequence convergence to $\tilde{\Xi}$ in $C([0,T],\mathbb{R}_\Omega)$ as $n\to\infty$. $\forall\,\tilde{\Xi}_n,\tilde{\Xi}\in C([0,T],\mathbb{R}_\Omega)$, for any $t\in[0,T]$, we get

$$\begin{aligned}\left|\Phi\left(\tilde{\Xi}_n(t,\theta)\right)-\Phi(\tilde{\Xi}(t,\theta))\right| \leq & \frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[|B(\lambda)|\left|\Im\left(\tilde{\Xi}_n(\lambda,\theta)\right)-\Im(\tilde{\Xi}(\lambda,\theta))\right| \\ & +\int_0^\lambda |\vartheta(\lambda,s)|\left|\Im\left(\tilde{\Xi}_n(s,\theta)\right)-\Im(\tilde{\Xi}(s,\theta))\right|ds \\ & +\int_0^T |\vartheta_1(\lambda,s)|\left|\Im\left(\tilde{\Xi}_n(s,\theta)\right)-\Im(\tilde{\Xi}(s,\theta))\right|ds\Big]d\lambda \\ & +\frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[|B(\lambda)|\left|\Im\left(\tilde{\Xi}_n(\lambda,\theta)\right)-\Im(\tilde{\Xi}(\lambda,\theta))\right| \\ & +\int_0^\lambda |\vartheta(\lambda,s)|\left|\Im\left(\tilde{\Xi}_n(s,\theta)\right)-\Im(\Xi(s,\theta))\right|ds \\ & +\int_0^T |\vartheta_1(\lambda,s)|\left|\Im\left(\tilde{\Xi}_n(s,\theta)\right)-\Im(\Xi(s,\theta))\right|ds\Big]d\lambda \\ \leq & \left(\frac{\Theta_2 LT^\eta}{\Gamma(\eta+1)}+\frac{\Theta_1 LT^{\eta+1}}{\Gamma(\eta+2)}+\frac{\Theta_1^* LT^{\eta+1}}{\Gamma(\eta+2)}\right)\left\|\tilde{\Xi}_n-\tilde{\Xi}\right\| \\ & \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Φ is implied to be continuous by this.

(ii) Next, we will demonstrate how $C([0,T],\mathbb{R}_\Omega)$ translates bounded set to itself using the operator Φ, i.e., $\forall\,\kappa>0,\exists$ a $n>0$ and $\forall\,\tilde{\Xi}\in E_\kappa$, we get $\|\Phi(\tilde{\Xi})\|\leq n$, and E_κ is defined by $E_\kappa=\left\{\tilde{\Xi}\in C([0,T],\mathbb{R}_\Omega):\|\tilde{\Xi}\|\leq\kappa\right\}$, $\forall\,t\in[0,T]$,

$$\begin{aligned}|\Phi(\tilde{\Xi}(t,\theta))| \leq & \frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[|B(\lambda)||\Im(\tilde{\Xi}(\lambda,\theta))| \\ & +\int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\Xi}(s,\theta))|ds+\int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\Xi}(s,\theta))|ds\Big]d\lambda \\ & +\frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1}\Big[|B(\lambda)||\Im(\tilde{\Xi}(\lambda,\theta))| \\ & +\int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\Xi}(s,\theta))|ds+\int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\Xi}(s,\theta))|ds\Big]d\lambda \\ \leq & \left(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)LT^\eta\|\tilde{\Xi}\| \\ \leq & \left(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)\kappa LT^\eta.\end{aligned}$$

By choosing $n=\left(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\right)\kappa LT^\eta$, we have $\|\Phi(\tilde{\Xi}(t,\theta))\|\leq n$. We get $\|\Phi\tilde{\Xi}\|\leq n$, the set E_k is implied to be confined by this.

(iii) Φ operators bounded set into equi-continuous sets of $C([0,T],\mathbb{R}_\Omega)$. Let $t_1,t_2\in(0,T]$ and $t_1<t_2$. $\forall\,\tilde{\Xi}\in E_\kappa$, we get

$$\begin{aligned}\left|\Phi\left(\tilde{\Xi}(t_2,\theta)\right)-\Phi\left(\tilde{\Xi}(t_1,\theta)\right)\right| =&\Big|\frac{1}{\Gamma(\eta)}\int_0^{t_2}(t_2-\lambda)^{\eta-1}\times\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta))\\
&+\int_0^{\lambda}\vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds+\int_0^{T}\vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda\\
&-\frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-\lambda)^{\eta-1}\times\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta))\\
&+\int_0^{\lambda}\vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds+\int_0^{T}\vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda\Big|\\
=&\Big|\frac{1}{\Gamma(\eta)}\int_0^{t_1}\left((t_2-\lambda)^{\eta-1}-(t_1-\lambda)^{\eta-1}\right)\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta))\\
&+\int_0^{\lambda}\vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds+\int_0^{T}\vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda\\
&+\frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-\lambda)^{\eta-1}\times\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta))\\
&+\int_0^{\lambda}\vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds+\int_0^{T}\vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda\Big|\\
\leq&\frac{\Theta_2L\|\tilde{\Xi}\|}{\Gamma(\eta+1)}\left|2(t_2-t_1)^{\eta}+(t_1^{\eta}-t_2^{\eta})\right|+\frac{\Theta_1L\|\tilde{\Xi}\|}{\Gamma(\eta+2)}\Big|2(t_2-t_1)^{\eta+1}\\
&+\left(t_1^{\eta+1}-t_2^{\eta+1}\right)\Big|+\frac{\Theta_1^*L\|\tilde{\Xi}\|}{\Gamma(\eta+2)}\Big|2(t_2-t_1)^{\eta+1}\\
&+\left(t_1^{\eta+1}-t_2^{\eta+1}\right)\Big|\\
&\longrightarrow 0 \text{ as } t_1\longrightarrow t_2.\end{aligned}$$

This illustrates how the operator converts $C([0,T],\mathbb{R}_\Omega)$ from a bounded set into an equicontinuous set.

Consequently, the Arzela-Ascoli theorem states that operator Φ is compact. For the final stage, let's use the set ι, which is given by

$$\iota=\left\{\tilde{\Xi}\in C([0,T],\mathbb{R}_\Omega):\tilde{\Xi}=\sigma\Phi(\tilde{\Xi})\text{ for }0<\sigma<1\right\}.$$

Now that the prior set is bounded, we may demonstrate it. Assume that $\tilde{\Xi}\in\iota$ and $\forall\,t\in[0,T]$, we get from

$$\begin{aligned}\tilde{\Xi}(t,\theta)=&\sigma\Big(\frac{1}{\Gamma(\eta)}\int_0^{T}(T-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta))+\int_0^{\lambda}\vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\\
&+\int_0^{T}\vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda+\frac{1}{\Gamma(\eta)}\int_0^{t}(t-\lambda)^{\eta-1}\times\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta))\\
&+\int_0^{\lambda}\vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds+\int_0^{T}\vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda\Big).\end{aligned}$$

Also, $|\tilde{\varphi}(t,\theta)|\leq\Theta_3$ from assumptions (H3), we get

Then, $\forall\, t \in [0,T]$, using $|\Im(\tilde{\Xi}(t,\theta))| \le \Theta^*$, we have

$$\begin{aligned}
|\tilde{\Xi}(t,\theta)| =& \Big|\sigma\Big(\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta)) + \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds \\
&+ \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1} \times \Big[B(\lambda)\Im(\tilde{\Xi}(\lambda,\theta)) \\
&+ \int_0^\lambda \vartheta(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds + \int_0^T \vartheta_1(\lambda,s)\Im(\tilde{\Xi}(s,\theta))ds\Big]d\lambda\Big)\Big| \\
\le& \Big|\frac{1}{\Gamma(\eta)}\int_0^T (T-\lambda)^{\eta-1}\Big[|B(\lambda)||\Im(\tilde{\Xi}(\lambda,\theta))| + \int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\Xi}(s,\theta))|ds \\
&+ \int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\Xi}(s,\theta))|ds\Big]d\lambda + \frac{1}{\Gamma(\eta)}\int_0^t (t-\lambda)^{\eta-1} \times \Big[B(\lambda)||\Im(\tilde{\Xi}(\lambda,\theta))| \\
&+ \int_0^\lambda |\vartheta(\lambda,s)||\Im(\tilde{\Xi}(s,\theta)) \mid ds + \int_0^T |\vartheta_1(\lambda,s)||\Im(\tilde{\Xi}(s,\theta)) \mid ds\Big]d\lambda \\
\le& \Big(\frac{\Theta_2(\eta+1) + (\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\Theta^* T^\eta\Big) \\
=& n^*,
\end{aligned}$$

where $n^* = \Big(\frac{\Theta_2(\eta+1)+(\Theta_1+\Theta_1^*)T}{\Gamma(\eta+2)}\Big)\Theta^* T^\eta$. This establishes the boundedness of every $\tilde{\Xi} \in \iota$. The set ι is hence limited. Additionally, Schaefer's theorem establishes the existence of a fixed point for the Φ. This indicates that there is at least one solution for the FFCV-FIDE (1)-(2), $\forall\, t \in [0,T]$, $\tilde{\Xi}(t)$. Moreover, we can demonstrate that FFCV-FIDE (1)-(2) has a unique continuous solution on $[0,T]$ by employing the assumptions in (H1)-(H3), and

$$\frac{(\Theta_2(\eta+1) + (\Theta_1+\Theta_1^*)T)}{\Gamma(\eta+2)} L T^\eta < 1,$$

is content. This may be inferred using the same method used to prove Theorem 3.1.

4 Methodology of ADT [4, 5]

This study demonstrates how to approximate the FFCV-FIDE solutions (1) using an ADT. Examine the FFCV-FIDE (1) that follows. The result of applying the I^η operator to both sides of the FFCV-FIDE (1) is

$$\begin{aligned}
\tilde{\Xi}(t,\theta) = \tilde{\Xi}_0 + I^\eta(\tilde{\varphi}(\lambda,\theta)) + I^\eta(B(t)\Im\tilde{\Xi}(t,\theta))\Big) \\
+ I^\eta\left[\int_0^t \vartheta(t,\lambda)\Im(\tilde{\Xi}(\lambda,\theta))d\lambda + \int_0^T \vartheta_1(t,\lambda)\Im(\tilde{\Xi}(\lambda,\theta))d\lambda\right].
\end{aligned}$$

ADT describes the solution $\tilde{\Xi}(t,\theta)$ as a series:

$$\tilde{\Xi}(t,\theta) = \sum_{i=0}^{\infty} \tilde{\Xi}_i(t,\theta), \tag{5}$$

and the nonlinear term M_1 are broken down as

$$M_1 = \sum_{i=0}^{\infty} P_i, \tag{6}$$

in which the Adomian polynomials P_i are supplied by

$$P_i = \frac{1}{i!}\frac{d^i}{dv^i}\left[M_1\left(\sum_{l=0}^{\infty} v^i \tilde{\Xi}_l\right)\right]_{v=0}.$$

Hence,

$$\begin{aligned}
P_0 &= M_1\left(\tilde{\Xi}_0\right),\\
P_1 &= \tilde{\Xi}_1 M_1'\left(\tilde{\Xi}_0\right),\\
P_2 &= \tilde{\Xi}_2 M_1'\left(\tilde{\Xi}_0\right) + \frac{1}{2}\tilde{\Xi}_1^2 M_1''\left(\tilde{\Xi}_0\right),\\
&\vdots
\end{aligned}$$

The components $\tilde{\Xi}_0, \tilde{\Xi}_2, \tilde{\Xi}_2, \ldots$ be found iteratively by

$$\begin{cases}
\tilde{\Xi}_0(t,\theta) = \tilde{\Xi}_0 + I^\eta(\tilde{\varphi}(\lambda,\theta)),\\
\tilde{\Xi}_1(t,\theta) = I^\eta\left(B(t)\tilde{\Xi}_0(t,\theta)\right) + I^\eta\left[\int_0^t \vartheta(t,\lambda)P_0 d\lambda + \int_0^T \vartheta_1(t,\lambda)P_0 d\lambda\right],\\
\vdots\\
\tilde{\Xi}_{k+1}(t,\theta) = I^\eta\left(B(t)\tilde{\Xi}_k(t,\theta)\right) + I^\eta\left[\int_0^t \vartheta(t,\lambda)P_k d\lambda + \int_0^T \vartheta_1(t,\lambda)P_k d\lambda\right].
\end{cases} \tag{7}$$

To approximate the IVP solution, we solve the aforementioned relation using the starting condition. The primary prerequisites for (1) are distinguished by their significant role in formulating the solution and their straightforward treatment of the recurrence connections. We construct this Adomian equation in such a way that the boundary condition is automatically satisfied by the final solution. If the series (5) is convergent uniformly, we can approximate the solution of FFCV-FIDE (1) by solving (7) and applying the initial condition, or by solving (7) and applying the initial condition, respectively, and receiving the M terms

$$\Phi_M(t,\theta) = \sum_{i=1}^{M-1} \tilde{\Xi}_i(t,\theta). \tag{8}$$

4.1 Convergence analysis

This section describes the convergence of the approximate solution for FFCV-FIDE that was previously discussed.

Theorem 4.1. *Let's assume that (H1)-(H3) are accurate. Take into consideration $0 < \Upsilon < 1$ as shown in (4). Subsequently, the series (5) converges uniformly to the FFCV-FIDE's solution $\tilde{\Xi}(t,\theta)$ in (1). Additionally, an approximate solution to $\tilde{\Xi}(t,\theta)$ is given by the partial sum (8).*

Proof. Keep in mind that since $\tilde{\varphi}(t,\theta) \in C(\Psi)$, $\tilde{\Xi}_0(t,\theta) \in C(\Psi)$. Thus, for any $t \in \Psi$, there exists $\Theta \in R$ and $\Theta > 0$ such that $|\tilde{\Xi}_0(t,\theta)| \leq \Theta$. We now demonstrate that the i-th term in the series (5) meets the given requirement.

$$|\tilde{\Xi}_i(t,\theta)| \leq \Theta\Upsilon^i \text{ on } \Psi, \tag{9}$$

where Υ was given in (4). For $i = 1$, we get

$$\begin{aligned}\left|\tilde{\Xi}_1(t,\theta)\right| =&\left|I^\eta\left[B(t)\Im\left(\tilde{\Xi}_0(t,\theta)\right) + \int_0^t \vartheta(t,\lambda)\Im\left(\tilde{\Xi}_0(\lambda,\theta)\right)d\lambda\right.\right.\\ &\left.\left.+\int_0^T \vartheta_1(t,\lambda)\Im\left(\tilde{\Xi}_0(\lambda,\theta)\right)d\lambda\right]\right|\\ \le&\Theta_2 L\left|\tilde{\Xi}_0(t,\theta)\right| I^\eta(1) + \Theta_1 L\left|\tilde{\Xi}_0(t,\theta)\right| I^\eta(t) + \Theta_1^* L\left|\tilde{\Xi}_0(t,\theta)\right| I^\eta(t)\\ =&\frac{\Theta_2 L}{\Gamma(\eta+1)}\left|\tilde{\Xi}_0(t,\theta)\right| t^\eta + \frac{\Theta_1 L}{\Gamma(\eta+2)}\left|\tilde{\Xi}_0(t,\theta)\right| t^{\eta+1} + \frac{\Theta_1^* L}{\Gamma(\eta+2)}\left|\tilde{\Xi}_0(t,\theta)\right| t^{\eta+1}\\ \le&\Upsilon\left|\tilde{\Xi}_0(t,\theta)\right| \le \Theta\Upsilon.\end{aligned} \tag{10}$$

Here, we consider (9) is true for $i = k-1$, i.e., $|\tilde{\Xi}_{k-1}(t,\theta)| \le \Theta\Upsilon^{k-1}$. Proceeding in the same way as previously, for $i = k$, we get

$$\begin{aligned}\left|\tilde{\Xi}_k(t,\theta)\right| =&\left|I^\eta\left[B(t)\Im\left(\tilde{\Xi}_{k-1}(t,\theta)\right) + \int_0^t \vartheta(t,\lambda)\Im\left(\tilde{\Xi}_{k-1}(\lambda,\theta)\right)d\lambda\right.\right.\\ &\left.\left.+\int_0^t \vartheta_1(t,\lambda)\Im\left(\tilde{\Xi}_{k-1}(\lambda,\theta)\right)d\lambda\right]\right|\\ \le&\Upsilon\left|\tilde{\Xi}_{k-1}(t,\theta)\right|\\ \le&\Theta\Upsilon^k.\end{aligned}$$

We find the required outcome at (9) as an answer. Consequently, for every $t \in \Psi$,

$$\sum_{i=0}^{\infty}\left|\tilde{\Xi}_i(t,\theta)\right| \le \sum_{i=0}^{\infty}\Theta\Upsilon^i, \tag{11}$$

where $\sum_{i=0}^{\infty}\Theta\Upsilon^i$ is a convergent series for $0 < \Upsilon < 1$. Then, the $\sum_{i=0}^{\infty}\tilde{\Xi}_i(t,\theta)$ is converges uniformly to the Weierstrass M-test. Therefore, again using the Weierstrass M-test. There is a uniform convergence of the series (5) on. As a result, the partial sum in (8) approximates the answer to (1). □

5 An Example

Example 1. Consider the following FFCV-FIDE:

$$\begin{cases} D^{\frac{1}{2}}\tilde{\Xi}(t,\theta) = \tilde{\varphi}(t,\theta) + B(t)\tilde{\Xi}(t,\theta) + \displaystyle\int_0^t \vartheta(t,\lambda)\tilde{\Xi}(\lambda,\theta)d\lambda + \int_0^1 \vartheta_1(t,\lambda)\tilde{\Xi}(\lambda,\theta)d\lambda, \quad t \in (0,1],\\ \tilde{\Xi}(0,\theta) = (\theta-1, 1-\theta),\end{cases}$$

and $\underline{\varphi}(t,\theta) = 3t(\theta-1), \bar{\varphi}(t,\theta) = 3t(1-\theta), B(t) = \frac{(-t^3)}{10}, \vartheta(t,\lambda) = \frac{-3\lambda t}{10}$ and $\vartheta_1(t,\lambda) = \frac{-6\lambda t}{10}$. The above equation's equivalent to

$$\begin{cases} D^{\frac{1}{2}}\underline{\Xi}(t,\theta) = 3t(\theta-1) - \dfrac{t^3}{10}\underline{\Xi}(t,\theta) - \displaystyle\int_0^t \frac{3\lambda t}{10}\underline{\Xi}(\lambda,\theta)d\lambda - \int_0^1 \frac{6\lambda t}{10}\underline{\Xi}(\lambda,\theta)d\lambda,\\ \underline{\Xi}(0,\theta) = (\theta-1),\\ D^{\frac{1}{2}}\bar{\Xi}(t,\theta) = 3t(\theta-1) - \dfrac{t^3}{10}\bar{\Xi}(t,\theta) - \displaystyle\int_0^t \frac{3\lambda t}{10}\bar{\Xi}(\lambda,\theta)d\lambda - \int_0^1 \frac{6\lambda t}{10}\bar{\Xi}(\lambda,\theta)d\lambda,\\ \bar{\Xi}(0,\theta) = (1-\theta).\end{cases}$$

Now let us construct $\underline{\Xi}(t,\theta)$ and apply $I^{\frac{1}{2}}$ on both sides:

$$\begin{aligned}\underline{\Xi}(t,\theta) =& \underline{\Xi}(0,\theta) + I^{1/2}(\underline{\varphi}(t,\theta)) + I^{1/2}(B(t)\underline{\Xi}(t,\theta)) + I^{1/2}(\vartheta(t,\lambda)\underline{\Xi}(\lambda,\theta)) + I^{1/2}(\vartheta_1(t,\lambda)\underline{\Xi}(\lambda,\theta)),\\ \underline{\Xi}(t,\theta) =& (\theta-1) + I^{1/2}(3t(\theta-1)) + I^{1/2}\left(\frac{-t^3}{10}(\theta-1)\right) + I^{1/2}\left(\frac{-3\lambda t}{10}(\theta-1)\right)\\ &+ I^{1/2}\left(\frac{-6\lambda t}{10}(\theta-1)\right).\end{aligned}$$

We are now using the ADT:

$$\begin{aligned}\underline{\Xi}_0(t,\theta) &= \underline{\Xi}(0,\theta) + I^{1/2}(3t(\theta-1)) = (\theta-1) + \left(\frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)}\right),\\ \underline{\Xi}_1(t,\theta) &= I^{1/2}\left(\frac{-(t^3)}{10t}(\theta-1)\right) + I^{1/2}\left(\frac{-3\lambda t}{10}(\theta-1)\right) + I^{1/2}\left(\frac{-6\lambda t}{10}(\theta-1)\right).\end{aligned}$$

In a similar manner, we may locate consecutive words and obtain the answer:

$$\underline{\Xi} = \sum_{n=0}^{\infty}\underline{\Xi}_n = (\theta-1) + \left(\frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)}\right) + \left(\frac{6t^{3/2}(\theta-1)}{\Gamma(5/2)}\right) + \ldots.$$

Similarly, we may locate

$$\overline{\Xi} = \sum_{n=0}^{\infty}\underline{\Xi}_n = (1-\theta) + \left(\frac{3t^{3/2}(\theta-1)}{\Gamma(5/2)}\right) + \left(\frac{6t^{3/2}(\theta-1)}{\Gamma(5/2)}\right) + \ldots.$$

Figure 1 illustrates fuzzy approximate solutions to varying values of θ, while Figure 2 shows fuzzy approximate solutions to varying values of θ and t.

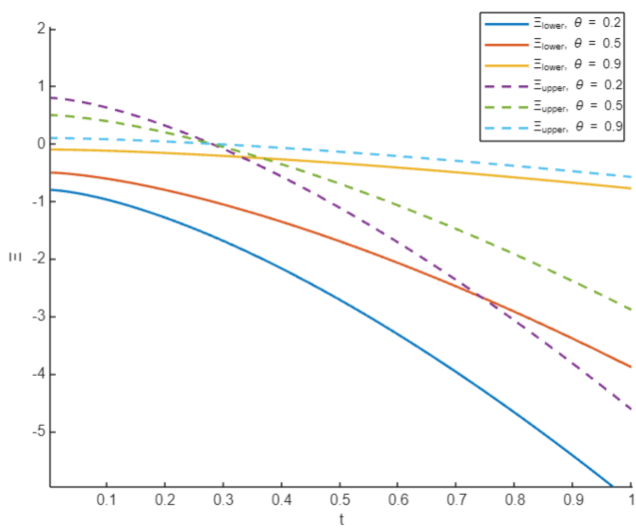

Figure 1: Fuzzy approximate solutions to varying values of θ.

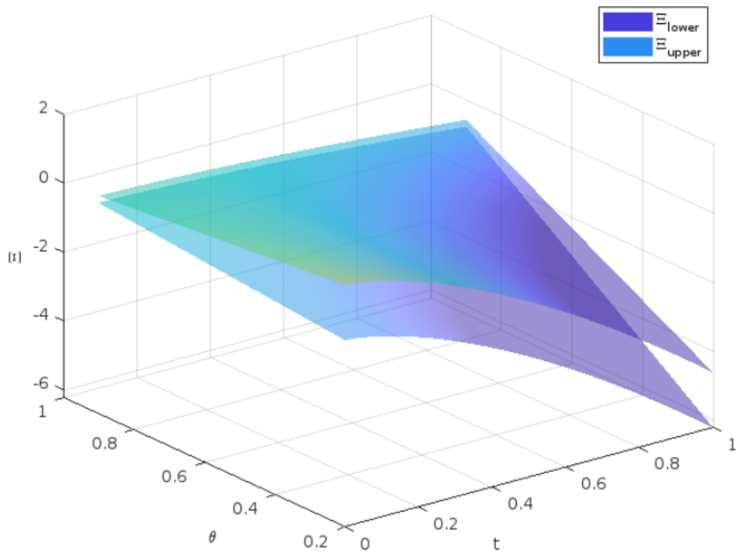

Figure 2: Fuzzy approximate solutions to varying values of θ and t.

6 Conclusions

This article examines the fractional derivative in the Caputo sense for the class of fuzzy Volterra-Fredholm integration equations of the first kind of fractional order. In this article, the initial value problems are considered simultaneously. The transformation from the first type to the second occurs according to Leibniz's rule. The fixed point theory is used to establish the existence and uniqueness of the equation under consideration in its second type. In addition, the Adomian decomposition method is used to determine the solution to the proposed problem. We provide some examples to support the approach. MATLAB is used to display numerical and graphical representations of the symmetry between the top and bottom layer representations of the fuzzy solutions. The obtained results have been validated by proving the appropriate problem. In the future, we extend our work with delay terms.

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